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# EQUATIONS OF BIFURCATION OF EQUILIBRIUM OF AN ELASTIC ISOTROPIC BODY IN TERMS OF RATES OF CHANGE OF LAGRANGEAN COORDINATES

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A new version of constructing the three-dimensional theory of elastic stability is advanced. Bifurcation is considered to be an interchange of material particles at a fixed point in space. As a kinematic variable we take the rate of change of Lagrangean particle coordinates. On the basis of obtained exact solutions, an approximate method is developed, valid in the case of small precritical deformations and rotations.

Whenever the usual Lagrangean presentation is used for the motion of a continuous medium [1, 2], the equations which determine the changes in the stress tensor necessarily contain the rotations of material particles. As a result, the linearized deformation equations of equilibrium in the general case contain the sought critical stresses [1]. It is of interest to study that version of a boundary value stability problem for which the parameters enter essentially only into the boundary conditions. One of these versions was suggested by Leibenzon [3], and then independently by Ishlinskii [4]. However, it cannot be obtained using the Lagrangean presentation while linearizing the original equations of the nonlinear theory of elasticity.

In the present paper, nonlinear equations of the theory of elasticity in Eulerian representation are obtained using as a kinematic variable the rates of change in Lagrangean coordinates. On the basis of these equations, bifurcation of equilibrium of an isotropic elastic body is considered. The advantage of the suggested version is that the changes in the Cauchy stress tensor are related only to the deformation tensor of introduced velocities. Therefore, the differential equations of equilibrium contain only those parameters of the prebifurcation state which are related to the change in physical properties of the body during deformation. The components of the rotation tensor of these velocities enter only into the boundary conditions in connection with the change of shape of the body at the instant of bifurcation. The parameters which enter as factors of the components of the rotation tensor in the boundary conditions are the most essential part of parameters of the precritical state which enter into the structure of the obtained boundary value problem for neutral equilibrium.

For an isotropic elastic body which is only slightly deformed in the precritical state, the indicated circumstance permits to suggest a simple approximate version of the equations of neutral equilibrium in which the sought parameter of the critical loading enters only into the boundary conditions. If the physical content of the sought functions which enter the differential equations and the boundary conditions is moved into the background, then it appears that the approximate version of the boundary value problem is close to the one used by Leibenzon.

**1.** We consider the slow motion of a deformable body in the case when the inertia forces can be disregarded. We use Cartesian coordinates with respect to the motionless space. The Lagrangean coordinates of the particles are defined by the coordinates $(a_1, a_2, a_3)$ of their positions in the space in the initial unstressed state of the body. In a current state, at an arbitrary instant, a fixed particle is to be found at the point of coordinates $x_i = x_i(a_1, a_2, a_3, t)$, $i = 1,2,3$. Then the equations $a_i = a_i(x_1, x_2, x_3, t)$ determine that particle which at the given instant is at the point $(x_1, x_2, x_3)$. For this point we introduce the quantity $v_i^* = -\partial a_i / \partial t$, which will be called the rate of change of the Lagrangean coordinates. Its connection with the velocity $v_i$ of the material particle is established with the aid of the identity

$$\frac{da_i}{dt} = \frac{\partial a_i}{\partial t} + v_m \frac{\partial a_i}{\partial x_m} = 0 \tag{1.1}$$

Consequently, at any instant,

$$v_i^* = v_m \frac{\partial a_i}{\partial x_m}, \qquad v_i = v_m^* \frac{\partial x_i}{\partial a_m}, \qquad v_m^* \frac{\partial}{\partial a_m} = v_m \frac{\partial}{\partial x_m} \tag{1.2}$$

In the sequel we will assume that $v_i^* = v_i^*(a_1, a_2, a_3, t)$. The above introduced rate of change of the Lagrangean coordinates allows us to investigate processes at a fixed point in space.

We consider the rate of change of the density $\rho$ of the medium at a current state, connected with the medium density $\rho^\circ$ at the initial state by the relation $\rho / \rho^\circ = \det \| \partial a_n / \partial x_m \|$. Differentiating this expression with respect to time for $x_i = \mathrm{const}$, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\rho}{\rho^\circ}\right) = A_{is} \frac{\partial}{\partial t}\left(\frac{\partial a_i}{\partial x_s}\right) = -A_{is} \frac{\partial v_i^*}{\partial x_s} = -A_{is} \frac{\partial a_m}{\partial x_s} \frac{\partial v_i^*}{\partial a_m} = -\frac{\rho}{\rho^\circ} \frac{\partial v_i^*}{\partial a_i} \tag{1.3}$$

Here $A_{is}$ is the cofactor of the element $\partial a_i / \partial x_s$ in the matrix $\| \partial a_n / \partial x_m \|$. If the medium density at the initial state is $\rho^\circ = \mathrm{const}$, then the following form of the continuity equation holds:

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial v_i^*}{\partial a_i} = 0 \tag{1.4}$$

As a measure of the deformation in the Euler representation of the motion, it is convenient to use the second Cauchy measure [6] $c_{ij} = (\partial a_m / \partial x_i)(\partial a_m / \partial x_j)$, connected with the Almansi strain tensor $\varepsilon_{ij}$ by the relation $c_{ij} = \delta_{ij} - 2\varepsilon_{ij}$, or the second Finger measure $f_{ij} = (\partial x_i / \partial a_m)(\partial x_j / \partial a_m)$. The tensors $c_{ij}$ and $f_{ij}$ are inverses of each other. We obtain the rate of change of these tensors at a fixed point of the space. First we compute

$$\begin{aligned}\frac{\partial c_{ij}}{\partial t} &= -\frac{\partial v_m^*}{\partial x_i}\frac{\partial a_m}{\partial x_j} - \frac{\partial a_m}{\partial x_i}\frac{\partial v_m^*}{\partial x_j} = -\frac{\partial v_m^*}{\partial a_n}\frac{\partial a_n}{\partial x_i}\frac{\partial a_m}{\partial x_j} - \\ &\frac{\partial v_m^*}{\partial a_n}\frac{\partial a_n}{\partial x_j}\frac{\partial a_m}{\partial x_i} = -\frac{\partial a_m}{\partial x_i}\frac{\partial a_n}{\partial x_j}\left(\frac{\partial v_m^*}{\partial a_n} + \frac{\partial v_n^*}{\partial a_m}\right) = \\ &-2e_{mn}^*\frac{\partial a_m}{\partial x_i}\frac{\partial a_n}{\partial x_j}, \qquad e_{mn}^* = \frac{1}{2}\left(\frac{\partial v_n^*}{\partial a_m} + \frac{\partial v_m^*}{\partial a_n}\right)\end{aligned} \tag{1.5}$$

Differentiating the relation $c_{im} f_{mj} = \delta_{ij}$, we get

$$\frac{\partial f_{ij}}{\partial t} = -f_{im}\frac{\partial c_{mn}}{\partial t} f_{nj} = 2e_{mn}^* \frac{\partial x_i}{\partial a_m}\frac{\partial x_j}{\partial a_n} \tag{1.6}$$

The tensor $e_{mn}^*$, introduced here, will be called the deformation tensor of the rate of change of the Lagrangean coordinates. We introduce also the rotation tensor of the velocities $v_i^*$

$$\overset{*}{\omega}_{mn} = \frac{1}{2}\left(\frac{\partial v_n^*}{\partial a_m} - \frac{\partial v_m^*}{\partial a_n}\right)$$

**2.** At an arbitrary instant, in the absence of body forces, we have, in the Eulerian representation, the following equilibrium equations inside the body and the boundary conditions at that part of the surface where the exterior forces are given:

$$\partial\sigma_{ij} / \partial x_i = 0, \qquad n_i\sigma_{ij} = F_j \tag{2.1}$$

Correspondingly, in the Lagrangean representation

$$\frac{\partial \pi_{ij}}{\partial a_i} = 0, \quad n_i^\circ \pi_{ij} = \frac{F_j dS}{dS^\circ} \quad \left(\sigma_{ij} = \frac{\rho}{\rho^\circ}\frac{\partial x_i}{\partial a_m}\pi_{mj}\right) \tag{2.2}$$

Here $\sigma_{ij}$ is the Cauchy stress tensor, $\pi_{ij}$ is the Piola stress tensor (these tensors are related through the equation given between the parentheses [6]), $F_j$ is the vector of the surface load corresponding to the area of the boundary surface in the current state, $dS$ and $dS^\circ$ are the elements of area on the boundary surfaces in the current and initial states, respectively, while $n_i$ and $n_i^\circ$ are the vectors of the unit normal to the boundary surface in the current and initial states, respectively, and [6]

$$n_i = n_m^\circ \frac{\partial a_m}{\partial x_i}\frac{\rho^\circ}{\rho}\frac{dS^\circ}{dS} \tag{2.3}$$

For a fixed point of the space the equilibrium equations expressed in the rates of change of the stress tensor [7]

$$\frac{\partial}{\partial x_i}\left(\frac{\partial\sigma_{ij}}{\partial t}\right) = 0 \tag{2.4}$$

The boundary conditions have to be related to a fixed material particle. Therefore, at an arbitrary instant we must have

$$\frac{dn_i}{dt}\sigma_{ij} + n_i\frac{d\sigma_{ij}}{dt} = \frac{dF_j}{dt}$$

Computing the derivative $dn_i / dt$ by differentiating (2.3) for $a_j = \text{const}$, we obtain the stress boundary conditions in terms of the rate of change of the stress tensor

$$n_i\left(\frac{\partial\sigma_{ij}}{\partial t} + v_m\frac{\partial\sigma_{ij}}{\partial x_m} - \frac{\partial v_i}{\partial x_m}\sigma_{mj} + \sigma_{ij}\frac{\partial v_m}{\partial x_m}\right)dS = \frac{d}{dt}(F_j dS) \tag{2.5}$$

The geometric boundary conditions are determined by the assignment of the velocities $v_i$.

**3.** It will be necessary to obtain the Cauchy and Piola stress tensors in terms of the Finger strain measure $f_{mn}$. In the case of a homogeneous, isotropic, perfectly elastic body, the specific potential energy $\Phi$, referred to the initial volume, is a function of three independent invariants of the first Cauchy measure [6]. The same invariants can be expressed also in terms of the components $f_{mn}$ of the second Finger strain measure. This allows us to assume that $\Phi = \Phi(f_{mn})$. Then, the necessary relations can be obtained starting from the expression [1]

$$\pi_{ij} = \frac{\partial \Phi}{\partial (\partial x_j / \partial a_i)} \tag{3.1}$$

Since $\Phi = \Phi(f_{mn})$, we have

$$\pi_{ij} = \frac{\partial \Phi}{\partial f_{mn}} \frac{\partial f_{mn}}{\partial (\partial x_j / \partial a_i)} = 2 \frac{\partial x_n}{\partial a_i} \frac{\partial \Phi}{\partial f_{nj}} \tag{3.2}$$

Applying the formula given in parentheses of (2.2), we find the representation of the Cauchy stress tensor

$$\sigma_{ij} = 2 \frac{\rho}{\rho^\circ} f_{in} \frac{\partial \Phi}{\partial f_{nj}} \tag{3.3}$$

It is significant that the Cauchy stress tensor depends only on the tensor components $f_{mn}$. A detailed discussion of the problem of the representation of the Cauchy stress tensor in terms of the strain measures $c_{mn}$ and $f_{mn}$ can be found in [8].

Making use of the relations (3.3), (1.3), (1.6), (3.2), we obtain the expression for the rate of change of the Cauchy stress tensor at a fixed point in space

$$\frac{\partial \sigma_{ij}}{\partial t} = \frac{\rho}{\rho^\circ} \frac{\partial x_i}{\partial a_m} (B_{mjst} e_{st}^* + 2 \overset{*}{e}_{ms} \pi_{sj} - \pi_{mj} e_{ss}^*) \tag{3.4}$$

$$B_{mjst} = 4 \frac{\partial^2 \Phi}{\partial f_{nj} \partial f_{pq}} \frac{\partial x_n}{\partial a_m} \frac{\partial x_p}{\partial a_s} \frac{\partial x_q}{\partial a_t} \tag{3.5}$$

The tensor $B_{mjst}$ determines the physical characteristics of the body in the deformation process. At the initial instant, for the undistorted body, the tensor $B_{mjst}$ is equal to the isotropic tensor ($\lambda$ and $\mu$ are the Lamé elastic constants)

$$D_{mjst} = \lambda \delta_{mj} \delta_{st} + \mu (\delta_{ms} \delta_{jt} + \delta_{mt} \delta_{js}) \tag{3.6}$$

A characteristic singularity of Eqs. (3.4) is the fact that only the components $e_{st}^*$ of the deformation tensor of the rate of change of the Lagrangean coordinates occur in them while the components $\omega_{st}^*$ of the rotation tensor do not occur.

**4.** On the basis of Eqs. (2.4), (2.5), (3.4) we derive a complete system of differential equations and boundary conditions relative to the vector $v_i^*$.

Substituting (3.4) into (2.5) we make use of the Piola identity

$$\frac{\partial}{\partial x_i} \left( \frac{\rho}{\rho^\circ} \frac{\partial x_i}{\partial a_m} \right) = 0 \tag{4.1}$$

which is given in a somewhat different form in [1]. Taking into account this identity, the differential equations of the equilibrium acquire the following form:

$$\frac{\partial}{\partial a_m} (B_{mjst} e_{st}^* + 2 \overset{*}{e}_{ms} \pi_{sj} - \pi_{mj} e_{ss}^*) = 0 \tag{4.2}$$

In order to convert the boundary conditions to the velocities $v_i^*$ we make use of the identity

$$n_i \left( v_m \frac{\partial \sigma_{ij}}{\partial x_m} - \frac{\partial v_i}{\partial x_m} \sigma_{mj} + \sigma_{ij} \frac{\partial v_m}{\partial x_m} \right) dS = n_i^\circ \left( v_m^* \frac{\partial \pi_{ij}}{\partial a_m} - \frac{\partial v_i^*}{\partial a_m} \pi_{mj} + \pi_{ij} \frac{\partial v_m^*}{\partial a_m} \right) dS^\circ \tag{4.3}$$

which can be verified in a straightforward manner. Taking into account this identity and the expressions (3.4), (2.3), the stress boundary conditions (2.5) obtain the form

$$n_m^\circ\left(B_{mjst}e_{st}^* + \frac{\partial v_s^*}{\partial a_m}\pi_{sj} + v_s^*\frac{\partial \pi_{mj}}{\partial a_s}\right)dS^\circ = \frac{d}{dt}(F_j dS) \tag{4.4}$$

At that part of the surface where the velocities $v_i$ of the material particles are given, the rate of change $v_i^*$ of the Lagrangean coordinates must be determined with the aid of the formulas (1.2).

The equations (4.2) can be transformed also into the form

$$\frac{\partial}{\partial a_m}\left(B_{mjst}e_{st}^* + \frac{\partial v_s^*}{\partial a_m}\pi_{sj} + v_s^*\frac{\partial \pi_{mj}}{\partial a_s}\right) = 0 \tag{4.5}$$

Equations (4.2) and (4.4) are the desired equations of the nonlinear elasticity theory in terms of the rates of change of the Lagrangean coordinates. In this case, the velocities $v_i^*$ occur linearly in them. The nonlinearity is determined by the expression (3.5) for the tensor $B_{mjst}$ and by the components $\pi_{st}$ of the Piola stress tensor.

For the solution of concrete problems we can use, for example, the method of successive loads. Knowing the rate of change of the Lagrangean coordinates, we can determine the displacement increments of the material particles, compute the tensor $B_{mjst}$ from (3.5), find the stresses $\pi_{st}$ and then determine again the rates for the next loading stage.

In the process of successive loads it may turn out that for some magnitude of the load the homogeneous boundary value problem for the velocities $v_i^*$ will have a nontrivial solution. This corresponds to the appearance of the characerisitc motion, i.e. to the equilibrium bifurcation.

**5.** In the case of potential ("dead") surface loads, the right-hand side of Eqs. (4.4) becomes equal to zero and the system of equations (4.2) and (4.4) obtains the form

$$\frac{\partial}{\partial a_m}(B_{mjst}e_{st}^* + 2\overset{*}{e}_{ms}\pi_{sj} - \pi_{mj}e_{ss}^*) = 0 \tag{5.1}$$

$$n_m^\circ\left(B_{mjst}e_{st}^* + \frac{\partial v_s^*}{\partial a_m}\pi_{sj} + v_s^*\frac{\partial \pi_{mj}}{\partial a_s}\right) = 0 \tag{5.2}$$

In order to obtain the homogeneous boundary value problem of neutral equilibrium, the bifurcation process will be considered as the appearance of the characteristic motion of the medium. The velocities of this motion will be denoted also by $v_i^*$. On the part of the boundary surface where the velocities of the material particles are given, the velocity of the characteristic motion has to be considered equal to zero. The parameters $B_{mjst}$ and $\pi_{st}$ for the prebifurcation state have to be determined, strictly speaking, from the solutions of the initial nonlinear equations. If these parameters are known, then the condition for the existence of a nontrivial solution of Eqs. (5.1) and (5.2) determines the critical state of the deformable body.

Equations (5.1) and (5.2) can be obtained as the Euler equation and the natural boundary conditions of some variational problem. We proceed with the known functional [9]

$$I = \int_{\tau_0}\frac{\partial^2\Phi}{\partial(\partial x_j/\partial a_i)\,\partial(\partial x_m/\partial a_n)}\frac{\partial v_j}{\partial a_i}\frac{\partial v_m}{\partial a_n}d\tau_0 \tag{5.3}$$

The condition for its stationary state leads to the equations of neutral equilibrium in the Lagrangean representation

$$\frac{\partial}{\partial a_i}\left(\frac{\partial^2 \Phi}{\partial (\partial x_j / \partial a_i) \partial (\partial x_m / \partial a_n)} \frac{\partial v_m}{\partial a_n}\right) = 0 \qquad (5.4)$$

$$n_i^\circ \frac{\partial^2 \Phi}{\partial (\partial x_j / \partial a_i) \partial (\partial x_m / \partial a_n)} \frac{\partial v_m}{\partial a_n} = 0$$

Substituting into (5.3) the relation which follows from (1.2)

$$\frac{\partial v_m}{\partial a_n} = \frac{\partial v_s^*}{\partial a_n} \frac{\partial x_m}{\partial a_s} + v_s^* \frac{\partial^2 x_m}{\partial a_s \partial a_n}$$

we obtain

$$I = \int_{\tau_0} \left(\frac{\partial^2 \Phi}{\partial (\partial x_j / \partial a_i) \partial (\partial x_m / \partial a_n)} \frac{\partial x_m}{\partial a_s} \frac{\partial v_s^*}{\partial a_n} + v_s^* \frac{\partial \pi_{ij}}{\partial a_s}\right) \frac{\partial v_j}{\partial a_i} d\tau_0$$

Making use now of (3.1) and (3.2), we arrive at the expression

$$I = \int_{\tau_0} \left(B_{ijmn} \overset{*}{e}_{mn} + \frac{\partial v_m^*}{\partial a_i} \pi_{mj} + v_m^* \frac{\partial \pi_{ij}}{\partial a_m}\right) \frac{\partial v_j}{\partial a_i} d\tau_0 \qquad (5.5)$$

The functional (5.5) is mixed, the velocities $v_m^*$ and $v_j$, connected by the relation (1.2), occur simultaneously in it. From the condition for the stationary state of the functional (5.5), we can obtain at once two versions of the boundary value problem for the neutral equilibrium: one in the form of Eqs. (5.4) and the other one in the form of Eqs. (4.5) and (5.2). In the case of the homogeneous initial deformed state, when $\partial^2 x_m / (\partial a_i \partial a_j) = 0$, the functional (5.5) is transformed to the form

$$I = \int_{\tau_0} \left(G_{ijmn} e_{mn}^* e_{ij}^* + t_{ij} \frac{\partial v_i^*}{\partial a_m} \frac{\partial v_j^*}{\partial a_m}\right) d\tau_0 \qquad (5.6)$$

$$G_{ijmn} = 4 \frac{\partial^2 \Phi}{\partial f_{st} \partial f_{pq}} \frac{\partial x_s}{\partial a_i} \frac{\partial x_t}{\partial a_j} \frac{\partial x_p}{\partial a_m} \frac{\partial x_q}{\partial a_n}, \qquad t_{ij} = 2 \frac{\partial \Phi}{\partial f_{st}} \frac{\partial x_s}{\partial a_i} \frac{\partial x_t}{\partial a_j}$$

Only the velocities $v_i^*$ occur in the functional (5.6).

**6.** The strict equations of neutral equilibrium (5.1), (5.2), obtained in terms of the rates of change of the Lagrangean coordinates, contain the components of the rotation tensor only in the boundary conditions. This opens the possibility of constructing approximate equations, in which the parameters of the precritical state of stress will not occur in the differential equations of the boundary value problem.

It is interesting to observe that such equations for homogeneous initial stresses, without any additional simplifying assumptions, are obtained by making use of the equation of state [6] ($\varepsilon_{ij}$ is the Almansi strain tensor)

$$\sigma_{ij} = \lambda \varepsilon_{ss} \delta_{ij} + 2\mu \varepsilon_{ij} \qquad (\varepsilon_{ss} = \varepsilon_{st} \delta_{ts}) \qquad (6.1)$$

Indeed, making use of (1.5) from (6.1), we obtain

$$\frac{\partial \sigma_{ij}}{\partial t} = \lambda \frac{\partial \varepsilon_{st}}{\partial t} \delta_{ts} \delta_{ij} + 2\mu \frac{\partial \varepsilon_{ij}}{\partial t} = \lambda \overset{*}{e}_{mn} \frac{\partial a_m}{\partial x_s} \frac{\partial a_n}{\partial x_t} \delta_{st} \delta_{ij} + \qquad (6.2)$$

$$2\mu \overset{*}{e}_{mn} \frac{\partial a_m}{\partial x_i} \frac{\partial a_n}{\partial x_j} = \lambda \overset{\circ}{f}_{mn} \overset{*}{e}_{mn} \delta_{ij} + 2\mu \overset{*}{e}_{mn} \frac{\partial a_m}{\partial x_i} \frac{\partial a_n}{\partial x_j}$$

where $f_{mn}^\circ = (\partial a_m / \partial x_s)(\partial a_n / \partial x_s)$ is Finger's first measure.

In order to simplify the writing in the equations which will follow, let us agree that no summation is to be performed with respect to two identical indices, one of which is in parentheses. Then, if the initial state of stress is homogeneous and the directions of the principal axes of the stress tensor coincide with the directions of the Cartesian coordinate axes $x_j$, then we can write $x_j = \lambda_{(j)} a_j$. Here $\lambda_j$ are the principal elongations. Switching in (6.2) to the variables $x_j$ and $v_j^{*\prime} = v_j^* / \lambda_{(j)}$, which corresponds to the use of the concomitant system of coordinates, we obtain

$$\frac{\partial \sigma_{ij}}{\partial t} = \lambda e_{ss}^{*\prime} \delta_{ij} + 2\mu e_{ij}^{*\prime}, \qquad e_{ij}^{*\prime} = \frac{1}{2}\left(\frac{\partial v_j^{*\prime}}{\partial x_i} + \frac{\partial v_i^{*\prime}}{\partial x_j}\right) \tag{6.3}$$

The relations (6.3) have the form of the usual Hooke's law in the linear theory of elasticity and do not contain explicitly the parameters of the precritical state of stress. In this case, Eqs. (2.4) obtain the form of the Lamé equations in the linear theory of elasticity

$$(\lambda + \mu) \frac{\partial}{\partial x_j}\left(\frac{\partial v_s^{*\prime}}{\partial x_s}\right) + \mu \frac{\partial^2 v_j^{*\prime}}{\partial x_s \partial x_s} = 0 \tag{6.4}$$

The corresponding homogeneous boundary conditions are given by (2.5), (1.2). They have the form

$$n_i \left(\lambda e_{ss}^{*\prime} \delta_{ij} + 2\mu e_{ij}^{*\prime} - \lambda_{(i)}^2 \sigma_{(j)} \frac{\partial v_i^{*\prime}}{\partial x_j} + \lambda_{(m)}^2 \frac{\partial v_m^{*\prime}}{\partial x_m} \sigma_{(i)} \delta_{ij}\right) = 0 \tag{6.5}$$

where $\sigma_i$ are the principal stresses. The equations (6.4), (6.5) form the boundary value problem of bifurcation in which the parameters of the precritical state of stress (the principal elongations and the principal stresses) occur only in the boundary conditions.

Let us clarify now under what simplifying assumptions we can obtain, for the boundary value stability problem, the same Lamé equations in the case of an ideal elastic body. For the sake of definiteness we take the specific potential energy in the form [1]

$$\Phi = {}^1/_2 \lambda I_1^2 + \mu I_2, \qquad I_1 = \overset{\circ}{\varepsilon}_{mn} \delta_{mn}, \qquad I_2 = \overset{\circ}{\varepsilon}_{mn} \overset{\circ}{\varepsilon}_{nm} \tag{6.6}$$

where $\varepsilon_{ij}^\circ$ is the Green strain tensor, connected with the first Cauchy measure $c_{ij}^\circ = (\partial x_m / \partial a_i)(\partial x_m / \partial a_j)$ by the relation $\varepsilon_{ij}^\circ = (c_{ij}^\circ - \delta_{ij}) / 2$. Making use of the equality of the invariants of the tensors $c_{ij}^\circ$ and $f_{ij}$, from (3.3) we obtain the equation of state

$$\sigma_{ij} = \frac{\rho}{\rho^\circ} [(\lambda I_1 - \mu) f_{ij} + \mu f_{im} f_{mj}] \tag{6.7}$$

where we take into account that

$$I_1 = (f_{mn} \delta_{mn} - 3) / 2 \tag{6.8}$$

Applying (1.3), (1.6) and the expression $\partial I_1 / \partial t = c_{st}^\circ e_{ts}^*$ which follows from (6.8), we obtain

$$\frac{\partial \sigma_{ij}}{\partial t} = - e_{ss}^* \sigma_{ij} + \frac{\rho}{\rho^\circ} \frac{\partial x_i}{\partial a_m} \frac{\partial x_j}{\partial a_n} [\lambda c_{st}^\circ e_{ts}^* \delta_{mn} + 2\mu c_{ms}^\circ e_{sn}^* + \tag{6.9}$$

$$2\mu e_{ms}^* c_{sn}^\circ + 2(\lambda I_1 - \mu) e_{mn}^*]$$

This equation can be written in the form (3.4), where now

$$B_{mjst} = [\lambda c_{st}^\circ \delta_{mn} + \mu (c_{ms}^\circ \delta_{nt} + c_{mt}^\circ \delta_{ns})] \frac{\partial x_j}{\partial a_n}$$

We introduce the energy stress tensor $\sigma_{ij}^\circ = \partial \varphi / \partial \varepsilon_{ij}^\circ$ [6] which, for the considered form (6.6) of the potential energy, is

$$\sigma_{ij}^{\circ} = \lambda \overset{\circ}{\varepsilon}_{mn}\delta_{ij} + 2\mu\varepsilon_{ij}^{\circ}$$

Then (6.9) can be written in the form

$$\frac{\partial\sigma_{ij}}{\partial t} = \frac{\rho}{\rho^{\circ}}\frac{\partial x_i}{\partial a_m}\frac{\partial x_j}{\partial a_n}[\lambda e_{ss}{}^*\delta_{mn} + 2\mu e_{mn}^* + 2\sigma_{ms}^{\circ}e_{sn}^* + 2e_{ms}^*\overset{\circ}{\sigma}_{sn} - \sigma_{mn}^{\circ}e_{ss}{}^* + 2\lambda(\varepsilon_{st}{}^{\circ}e_{st}{}^*\delta_{mn} - \overset{\circ}{\varepsilon}_{ss}e_{mn}^*)] \tag{6.10}$$

We consider now the case when the critical stresses at the loss of stability are considerably smaller than the elasticity moduli and the critical strains are much smaller than unity. Such an assumption holds, for example, for metal building structures. Then the first two terms which occur in brackets in (6.10), with the factors $\lambda$, $\mu$, are substantially greater than the remaining terms which determine the deformation anisotropy of the elastic body. In this case, in the expression (6.10) it is admissible to neglect those terms which are the products of the components of the initial stresses and strains by the components of the strain tensor $e_{st}{}^*$, which is equivalent to the disregard of the deformation anisotropy of the elastic body. In this case the expression (6.10) becomes

$$\frac{\partial\sigma_{ij}}{\partial t} = \frac{\rho}{\rho^{\circ}}\frac{\partial x_i}{\partial a_m}\frac{\partial x_j}{\partial a_n}(\lambda e_{ss}{}^*\delta_{mn} + 2\mu e_{mn}^*) \tag{6.11}$$

The equilibrium equations (2.5), taking into account the identity (4.1), obtain the form

$$\frac{\partial}{\partial a_m}\left[(\lambda e_{ss}{}^*\delta_{mn} + 2\mu e_{mn}^*)\frac{\partial x_j}{\partial a_n}\right] = 0 \tag{6.12}$$

If, furthermore, the initial rotations are also small, then $\partial x_j / \partial a_n \approx \delta_{nj}$ and instead of (6.12) the following equations are satisfied:

$$\frac{\partial}{\partial a_m}(\lambda e_{ss}{}^*\delta_{mn} + 2\mu e_{mn}^*) = 0 \tag{6.13}$$

The corresponding boundary conditions follow from (2.3), (2.5), (4.3)

$$n_m{}^{\circ}\left(\lambda e_{ss}{}^*\delta_{mn} + 2\mu e_{mn}^* - e_{ms}^*\pi_{sn} + \omega_{ms}^*\pi_{sn} + v_s{}^*\frac{\partial\pi_{mn}}{\partial a_s}\right) = 0 \tag{6.14}$$

Here we have made use also of the equality $\partial v_m{}^* / \partial a_s = e_{ms}{}^* - \omega_{ms}{}^*$.

The Piola tensor of precritical stresses in (6.14) has to be considered equal to the stress tensor defined with the aid of the linear elasticity theory, because the initial strains and rotations are assumed to be small. With the same degree of accuracy which has been used for the passage from (6.10) to (6.11), one can neglect the terms $e_{ms}{}^*\pi_{sn}$ in the equations (6.14). Then the boundary conditions become

$$n_m{}^{\circ}\left(\lambda e_{ss}{}^*\delta_{mn} + 2\mu e_{mn}^* + \omega_{ms}^*\pi_{sn} + v_s{}^*\frac{\partial\pi_{mn}}{\partial a_s}\right) = 0 \tag{6.15}$$

Equations (6.13) can be written in the form of the Lame equations

$$(\lambda + \mu)\frac{\partial}{\partial a_n}\left(\frac{\partial v_m{}^*}{\partial a_m}\right) + \mu\frac{\partial^2 v_n{}^*}{\partial a_m \partial a_m} = 0 \tag{6.16}$$

Thus, for the approximate solution of the stability problems of weakly deformable elastic bodies, we can make use of the boundary value problem (6.15), (6.16), with a parameter in the boundary conditions. For its derivation we have made assumptions on the possibility of disregarding the precritical strains and rotations, which corresponds to such a formulation of the stability problem where in the precritical state the elastic body is considered to be under tension without being deformed.

The boundary value problem described by Eqs. (6.14), (6.15) is similar to the corresponding boundary value problem in Leibenzon's method [3]. If we digress from the interpretation of the unknown functions which occur in the equations, then these problems differ only in some unimportant terms in the boundary conditions. Previously, one of the authors of the present paper has determined the rightfulness of the application of Leibenzon's method to the approximation solution of the stability problems of weakly deformable isotropic elastic bodies (*).

The equations (6.15), (6.16) are not the natural boundary conditions and the Euler equations of some functional similar to (5.6). However, in the opinion of the authors, for the determination of the first smallest eigenvalue of the boundary value problem, this circumstance is immaterial. At the same time, the application of the obtained approximate equations can significantly facilitate the solution of concrete problems, especially in those cases when simple solutions of the homogeneous Lamé equations are known.

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